International Journal of Theoretical Physics, Vol. 42, No. 1, January 2003 (© 2003)

Chaotic Maps, Invariants, Bose Operators, and Coherent States

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Received September 13, 2002

We first show how an autonomous system of ordinary first-order difference equations can be embedded into a Hilbert space description by using Bose operators and coherent states. Then we describe how an invariant can be expressed using Bose operators. Two examples are given.

KEY WORDS: Bose operators; coherent states; nonlinear difference equation; linearization.

It is well known that nonlinear ordinary and nonlinear partial differential equations can be embedded into linear equations in Hilbert space by using Bose operators and Bose field operators, respectively (Kowalski and Steeb, 1991). Here we consider an autonomous system of (nonlinear) difference equations and embedding. We describe how invariants can be expressed as Bose operators. Consider the autonomous system of first-order ordinary difference equations

$$x_{1,\,t+1} = f_1(x_{1,t}, x_{2,t}), \quad x_{2,\,t+1} = f_2(x_{1,t}, x_{2,t}), \qquad t = 0, 1, 2, \ldots \tag{1}$$

where we assume that f_1 and f_2 are analytic functions and $x_{1,0}$, $x_{2,0}$ are the initial values with $x_{1,t}, x_{2,t} \in \mathbf{R}$. The extension to higher dimensions is straightforward. To embed this system into a Hilbert space by using Bose operators $b_j^\dagger, b_j$ with $j = 1, 2$, we consider the Hilbert space states (Kowalski and Steeb, 1991)

$$|x_1, x_2, t\rangle := \exp\left(\frac{1}{2}\left(x_{1,t}^2 + x_{2,t}^2 - x_{1,0}^2 - x_{2,0}^2\right)\right)|x_{1,t}, x_{2,t}\rangle, \tag{2}$$

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where $|x_{1,t}, x_{2,t}\rangle$ is the normalized coherent state

$$|x_{1,t}, x_{2,t}\rangle := \exp\left(-\frac{1}{2}\left(x_{1,t}^2 + x_{2,t}^2\right)\right)\exp(x_{1,t}b_1^\dagger + x_{2,t}b_2^\dagger)|0\rangle, \tag{3}$$

where $|0\rangle$ denotes the vacuum state with $\langle 0|0\rangle = 1$ and $b_j|0\rangle = 0$. We recall that the Bose operators satisfy the commutation relation

$$[b_j, b_k] = [b_j^\dagger, b_k^\dagger] = 0, \qquad [b_j, b_k^\dagger] = \delta_{jk}I, \tag{4}$$

where $j, k = 1, 2$, and I is the identity operator. Next we introduce the evolution operator

$$\hat{M} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{b_1^{\dagger j}}{j!}\frac{b_2^{\dagger k}}{k!}(f_1(b_1, b_2) - b_1)^j(f_2(b_1, b_2) - b_2)^k. \tag{5}$$

It follows that

$$|x_1, x_2, t+1\rangle = \hat{M}|x_1, x_2, t\rangle, \qquad t = 0, 1, 2, \ldots \tag{6}$$

Thus the system of nonlinear difference equations (1) is mapped into a linear difference Eq. (6) in a Hilbert space. The price to be paid for linearity is that we have to deal with Bose operators which are linear unbounded operators. Furthermore we have the eigenvalue equations

$$b_1|x_1, x_2, t\rangle = x_{1,t}|x_1, x_2, t\rangle, \qquad b_2|x_1, x_2, t\rangle = x_{2,t}|x_1, x_2, t\rangle, \tag{7}$$

for the states given by (2), since $|x_{1,t}, x_{2,t}\rangle$ is a coherent state. Let $K(x_1, x_2)$ be an analytic function of x_1, x_2. Let $\hat{K}(b_1, b_2)$ be the corresponding operator. Then using (7), we have

$$\hat{K}(b_1, b_2)|x_1, x_2, t\rangle = K(x_{1,t}, x_{2,t})|x_1, x_2, t\rangle. \tag{8}$$

Thus

$$[\hat{K}, \hat{M}]|x_1, x_2, t\rangle = (K(x_{1,t+1}, x_{2,t+1}) - K(x_{1,t}, x_{2,t}))|x_1, x_2, t+1\rangle, \tag{9}$$

where $[\hat{K}, \hat{M}] = \hat{K}\hat{M} - \hat{M}\hat{K}$. Thus $\hat{K}$ is an invariant, i.e.,

$$K(x_{1,t+1}, x_{2,t+1}) = K(x_{1,t}, x_{2,t}), \tag{10}$$

if

$$[\hat{K}, \hat{M}] = 0. \tag{11}$$

For the actual calculation of the commutators, we use the formula

$$[f_i(\mathbf{b}), b_j^\dagger] = \frac{\partial}{\partial b_j} f_i(\mathbf{b}), \tag{12.a}$$

$$[b_i, g_j(\mathbf{b}^\dagger)] = \frac{\partial}{\partial b_j^\dagger} g_j(\mathbf{b}^\dagger), \tag{12.b}$$

and

$$[AB, C] = A[B, C] + [A, C]B, \qquad [A, BC] = [A, B]C + B[A, C]. \tag{13}$$

As our first example, let us consider the logistic equation

$$x_{t+1} = 2x_t^2 - 1, \qquad t = 0, 1, 2, \ldots, \qquad x_0 \in [-1, 1], \tag{14}$$

which is the most studied equation with chaotic behavior. All quantities of interest in chaotic dynamics can be calculated exactly. Examples are the fixed points and their stability, the periodic orbits and their stability, the moments, the invariant density, the topological entropy, the metric entropy, the Lyapunov exponent, and the autocorrelation function. The exact solution of (14) takes the form

$$x_t = \cos(2^t \arccos(x_0)), \tag{15}$$

since $\cos(2\alpha) \equiv 2\,\cos^2(\alpha) - 1$. The Lyapunov exponent for almost all initial conditions is given by $\ln(2)$. The logistic equation is an invariant of a class of second-order difference equations

$$x_{t+2} = g(x_t, x_{t+1}), \qquad t = 0, 1, 2, \ldots. \tag{16}$$

This means that if (14) is satisfied for a pair (x_t, x_{t+1}), then (16) implies that (x_{t+1}, x_{t+2}) also satisfies (14). In general, let

$$x_{t+1} = f(x_t), \qquad t = 0, 1, 2, \ldots \tag{17}$$

be a first-order difference equation. Then (17) is called an invariant of (16) if

$$g(x, f(x)) = f(f(x)). \tag{18}$$

We find that the logistic map (14) is an invariant of the trace map (Steeb, 2002)

$$x_{t+2} = 1 + 4x_t^2(x_{t+1} - 1). \tag{19}$$

The trace map plays an important role for the study of tight-binding Schrödinger equations with disorder. The second-order difference Eq. (19) can be written as a first-order system of difference equations $(x_{1,t} \equiv x_t)$, $x_{2,t} \equiv x_{t+1}$,

$$x_{1,t+1} = x_{2,t}, \qquad x_{2,t+1} = g(x_{1,t}, x_{2,t}). \tag{20}$$

After embedding the two maps into the linear unbounded operators $\hat{M}$ and $\hat{K}$, we can show that $[\hat{M}, \hat{K}] = 0$ using the commutation relation given above.

Another example is the Fibonacci trace map (Steeb, 2002)

$$x_{t+3} = 2x_{t+2}x_{t+1} - x_t. \tag{21}$$

This map admits the invariant

$$I(x_t, x_{t+1}, x_{t+2}) = x_t^2 + x_{t+1}^2 + x_{t+2}^2 - 2x_t x_{t+1} x_{t+2} - 1. \tag{22}$$

The Fibonacci trace map can be written as a system of three first-order difference equations. After embedding the two maps into the linear unbounded operators $\hat{M}$ and $\hat{K}$, we find that $[\hat{M}, \hat{K}] = 0$, since I is an invariant of (21).

REFERENCES

Kowalski K. and Steeb, W.-H. (1991). *Nonlinear Dynamical Systems and Carleman Linearization*, World Scientific, Singapore.

Steeb, W.-H. (2002). *Problems and Solutions in Theoretical and Mathematical Physics*. *Vol. II: Advanced Level*, 2nd edn., World Scientific, Singapore.